Electric Field Computations using Axial Green Function Method on Refined Axial Lines

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Axial Green function Methods (AGM) use so-called axial lines which are parallel to axes. These novel approaches can have computational advantages particularly in complicated domains. The attention here is laid on the computations of the electric potential problems with singularities caused by domain geometry and boundary values. In the vicinity of the singularities, we select singular subdomains on which the axial lines are independently refined. Based on these refinements of axial lines in AGM, we can accurately calculate the singular solutions.

***Index Terms*—Axial Green function Methods, Complicated domains, Refined axial lines, Singular solutions, Subdomains.**

I. Introduction

By the axial Green function, we mean that it is 1D Green function of an ordinary differential operator defined on a line parallel to axis in 2D/3D domain. In general, the finite difference method(FDM) uses this kind of lines, called the grids, but the admissible grids in FDM are very restrictive unless the domain is simple or the grids are gradually changing in space. Axial Green function methods(AGM) we have developed in [1], [2], [3] work well in arbitrary domains without deterioration of accuracy, and moreover they do even in randomly spacing axial lines. Traditionally, the use of Green function takes place in the boundary element(BEM) method, which can reduce the dimension of the problem by discretizing the boundary of the domain. However, this is possible only when finding the fundamental solution or Green function of 2D/3D differential operator. However, if the material coefficient is a function, then the BEM suffers from finding this multi-dimensional Green function in the domain. The advantages of AGMs are obvious at least in two points: (a) arbitrarily distributed axial lines are available, which is inconvenient in FDMs, and (b) it is much easier to find the 1D Green function compared to finding 2D/3D Green function. However, when singularities happen in the solution triggered by the geometric factor and/or the boundary values, the accumulation or refinement of axial lines near the singularity causes redundant axial lines in other places of the domain, which can often make a burden for computations using AGMs. According to these facts, the refinement of axial lines in the vicinity of singularity is essential, which is the key difference from the previous works. The selected region near the singularity is called the subdomain of singularity, on which axial lines can be independently distributed. The success of this approach is fully attributed to the representation formula of the solution on axial lines by the axial Green functions. Furthermore, it should be mentioned that AGMs have analytic formula for the derivative calculation of the solution. It often becomes troublesome in FEMs.

II. Axial Green function method

We consider the electric field problem in 2D domain Ω:

$$-\nabla \cdot (\epsilon \nabla u) = f, \quad \text{in } \Omega, \tag{1}$$

$$u = u^{\partial\Omega}, \quad \text{on } \partial\Omega, \tag{2}$$

where $\epsilon(x, y)$ is the permittivity of the matter. Of interest is

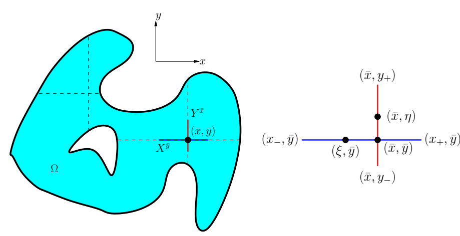

Fig. 1. Local axial lines in a domain Ω: $x-$axial line($X^{\bar{y}}$) and $y-$axial line($Y^{\bar{x}}$) at $(\bar{x}, \bar{y})$.

that this 2D problem can be reformulated to a couple of 1D problems. First of all, we decompose the 2D partial differential differential operator $-\nabla \cdot (\epsilon \nabla)$ into two parts by introducing a new variable $\phi(x, y)$ as follows:

$$-(\epsilon\, u_x)_x = \phi, \quad \text{in } \Omega, \tag{3}$$

$$-(\epsilon\, u_y)_y = f - \phi, \quad \text{in } \Omega. \tag{4}$$

Corresponding to the differential equations in (3) and (4), we find 1D Green functions, $G(x, \xi; X^{\bar{y}})$ and $G(y, \eta; Y^{\bar{x}})$, respectively, to represent the solution $u(x, y)$ on $x-$axial line $X^{\bar{y}}$ and $y-$axial line $Y^{\bar{x}}$ associated with a given cross point $(\bar{x}, \bar{y}) \in \Omega$ as shown in Fig. 1: for $x_- < \xi < x_+$ and $y_- < \eta < y_+$,

$$u(\xi, \bar{y}) = \int_{X^{\bar{y}}} G(x, \xi; X^{\bar{y}}) \phi(x, \bar{y})\, dx + u(x_-, \bar{y}) B_-^X(\xi) + u(x_+, \bar{y}) B_+^X(\xi), \tag{5}$$

$$u(\bar{x}, \eta) = \int_{Y^{\bar{x}}} G(y, \eta; Y^{\bar{x}}) (f - \phi)(\bar{x}, y)\, dy + u(\bar{x}, y_-) B_-^Y(\eta) + u(\bar{x}, y_+) B_+^Y(\eta). \tag{6}$$

These representations of the solution u on axial lines can be written as a unified 1D form: for $t_- < \tau < t_+$,

$$u(\tau) = \int_{t_-}^{t_+} G(t,\tau)g(t)\,dt + u(t_-)B_-(\tau) + u(t_+)B_+(\tau), \quad (7)$$

where $G(t,\tau)$ is the corresponding Green function in [3] and $B_\pm(\tau)$ is the function related to the boundary values $u(t_\pm)$. Instead of directly attacking the 2D problem in (1) with boundary condition (2), we pay attention to the equations of integral form in (5) and (6).

III. Refinement of Axial lines

For more accurate computation of the electric field in a given domain, we need independent refinements of axial lines on the selected subdomains on which singular behaviors happen. Assume that Ω is the given domain as illustrated in Fig. 2 and the boundary values are assigned on its boundary, where Neumann boundary conditions are available in AGM [4]. There happen three singularities of the solution u which comes from (**A**) the geometric singularity at $(0,0)$, (**B**) the different types of boundary condition at $(-1,0)$, and (**C**) the discontinuous boundary values at $(1,1)$. As seen in Fig. 2, we select three subdomains for the refinement of axial lines in the vicinity of these singular points. The axial lines on subdomains can be independently constructed no matter how the background axial lines are distributed. In our case, the accumulated axial lines near the singularity are taken for the purpose of refinement on all subdomains in the same pattern as in Fig. 2. The

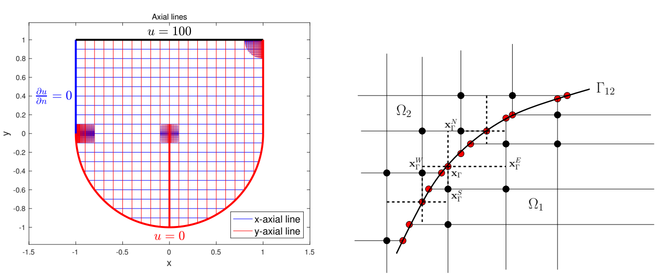

Fig. 2. Configuration of the problem domain Ω: [Left] three subdomains of singularities with refined axial lines, the background axial lines, and the boundary conditions. [Right] the interface point $\mathbf{x}_\Gamma$ (red dot) between a subdomain Ω_1 and the background Ω_2, and virtual axial lines of $\mathbf{x}_\Gamma$ (dotted line).

solution representations in (5) and (6) enable us to glue the AGM solutions on the subdomains to the background solution across the interface point $\mathbf{x}_\Gamma$ by considering virtual axial lines (dotted line) in the right panel of Fig. 2. Indeed, if $(\bar{x}, \bar{y})$ is the cross point (black dot), then the solid lines are taken as axial lines, $X^{\bar{y}}$ and $Y^{\bar{x}}$, in (5) and (6), as in usual AGMs, and then we replace $\xi = \bar{x}$ and $\eta = \bar{y}$. At the interface point $\mathbf{x}_\Gamma = (x_\Gamma, y_\Gamma)$ in Fig. 2, we choose the dotted lines passing through $\mathbf{x}_\Gamma$ as virtual axial lines, X^{y_Γ} and Y^{x_Γ}, and follow the same process as with the prior. Discretizing these integral equations for the unknown ϕ and u, we can solve the resultant system of equations using the generalized minimal residual method(GMRES). In Fig. 3 and 4, we calculate the electric potentials u and the electric field strengths $|\nabla u|$ in cases where

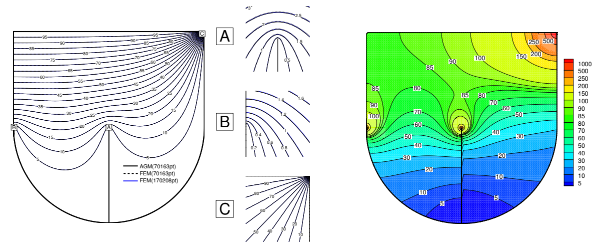

Fig. 3. Constant permittivity($\epsilon(x,y) = 1$): [Left] AGM solution u of $70,163$ cross points(black line) comparing to FEM solutions of $70,163$ nodes (dotted line) and $170,208$ nodes (blue line). [Right] AGM gradient field strength $|\nabla u|$.

$\epsilon(x,y) = 1$ and $\epsilon(x,y) = 5.5 + 3\tan^{-1}(-50(r-0.2))$ in Ω, respectively. In the left panels of Fig. 3 and 4, we show

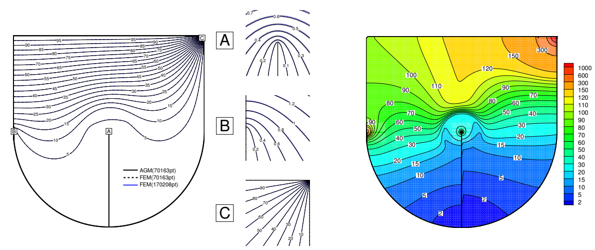

Fig. 4. Variable permittivity($\epsilon(x,y) = 5.5 + 3\tan^{-1}(-50(r-0.2))$): all the same setting as in Fig. 3.

two FEM solutions of $170,208$ nodes and $70,163$ nodes and the AGM solution of $70,163$ cross points, and the zoomed-in views on three subdomains of (**A**), (**B**), and (**C**) for comparison. The FEM meshes are generated using the AGM cross points in the left panel of Fig. 2. The strengths of AGM electric field ∇u are depicted in the right panels of Fig. 3 and 4. A particular emphasis in AGM is the derivative calculation, which is basically done from direct differentiation of (7).

IV. Conclusions

We present a refinement approach for the electric field computation using AGM, which is inevitable to calculate the accurate solution near singularity. Any type of subdomain can be taken for the singularity and arbitrary distribution of axial lines is available on that subdomain. The derivative computation using the solution representation formula on axial lines is superior to other methods.

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